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Physically Realistic Solutions to the Ernst Equation on Hyperelliptic Riemann Surfaces

C. Klein

Institut für Theoretische Physik, Universität Tübingen, Auf der Morgenstelle 14, 72076 Tübingen, Germany

O. Richter

Sektion Physik der Universität München, Theresienstraße 37, 80333 München, Germany

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Abstract

We show that the class of hyperelliptic solutions to the Ernst equation (the stationary axisymmetric Einstein equations in vacuum) previously discovered by Korotkin and Neugebauer and Meinel can be derived via Riemann–Hilbert techniques. The present paper extends the discussion of the physical properties of these solutions that was begun in a Physical Review Letter, and supplies complete proofs. We identify a physically interesting subclass where the Ernst potential is everywhere regular except at a closed surface which might be identified with the surface of a body of revolution. The corresponding spacetimes are asymptotically flat and equatorially symmetric. This suggests that they could describe the exterior of an isolated body, for instance a relativistic star or a galaxy. Within this class, one has the freedom to specify a real function and a set of complex parameters which can possibly be used to solve certain boundary value problems for the Ernst equation. The solutions can have ergoregions, a Minkowskian limit and an ultrarelativistic limit where the metric approaches the extreme Kerr solution. We give explicit formulae for the potential on the axis and in the equatorial plane where the expressions simplify. Special attention is paid to the simplest non–static solutions (which are of genus two) to which the rigidly rotating dust disk belongs.

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1 Introduction

It is generally believed that most of the stars and galaxies can be described in good approximation as fluid bodies in thermodynamical equilibrium. In the framework of general relativity, this implies (see e.g. [1, 2]) that the corresponding spacetimes are stationary and axisymmetric. Moreover it is usually assumed (though there is no proof known to us) that they are equatorially symmetric. This stresses the importance of the study of stationary axisymmetric spacetimes. A relativistic treatment is necessary for rapidly rotating and massive compact objects like pulsars, neutron stars and black-holes.

Though the importance of global solutions describing stationary axisymmetric fluid bodies is generally accepted, the complicated structure of the Einstein equations with matter gives little hope that such solutions can be found in the near future. Only for special and somewhat unphysical equations of state [3, 4, 5], it was possible to give solutions in the matter region which are discussed as candidates for an interior solution. In the exterior vacuum region, however, powerful solution generating techniques are at hand. Since the surface Γ_z of a compact astrophysical object constitutes a natural boundary at which the metric functions are not continuously differentiable, one is looking for solutions to the vacuum equations that are analytic outside this contour and can be at least continuously extended to Γ_z. This means that the typical problem one has to consider for the vacuum Einstein equations is a boundary value problem of Dirichlet, von Neumann or mixed type, see [6]. The matter then enters only in form of boundary conditions for the vacuum equations. This is possible if an interior solution is known or if only two–dimensionally extended bodies like disks or shells are considered. In the latter case, the surfacelike distribution of the matter implies that the matter equations reduce to ordinary differential equations. Notice that disks are important models in astrophysics for certain types of galaxies.

The reason why it is much more promising to treat only the vacuum case is the equivalence of the stationary axisymmetric Einstein equations to a single nonlinear differential equation for a complex potential, the so called Ernst equation [7]. The latter belongs to a family of completely integrable nonlinear equations that are studied as the integrability conditions for associated linear differential systems. The common feature of these linear systems is that they contain an additional variable, the so called spectral parameter, which reflects an underlying symmetry of the differential equations under investigation, in the case of the Ernst equation the Geroch group [8]. Associated linear systems for the Ernst equations were given in [9, 10, 11]. The existence of this parameter can be used to construct solutions by prescribing the singular structure of the matrix of the linear system with respect to the spectral parameter.

One of the most successful solution techniques for nonlinear differential equations rests on methods of algebraic geometry and leads to the so called finite-gap solutions that can be expressed elegantly in terms of theta functions. Such methods were first used to construct periodic and quasiperiodic solutions to nonlinear evolution equations like the Korteweg–de Vries (KdV) and the Sine–Gordon (SG) equation. For a survey of this subject we refer the reader to [12, 13]. However, it was only recently that algebro-geometrical methods were applied to the Ernst equation, see [14]. The found solutions differ from similar solutions of other equations in several aspects, e.g. they are in general not periodic or quasi-periodic. The main difference is that this class is much richer than previously obtained ones.

The development of solution techniques yields a deeper insight into the structure of nonlinear differential equations. However, from a practical point of view, it would also be desirable to solve initial value problems or, for the Ernst equation, boundary value problems. One approach to solve boundary value problems of the above mentioned type with the help of the linear system is to translate the physical boundary conditions into a Riemann-Hilbert problem which is equivalent to a linear integral equation, see [12]. Neugebauer and Meinel [15] succeeded in doing this in the case of the rigidly rotating dust disk. They were able to reduce the matrix problem on a sphere to a scalar Riemann-Hilbert problem on a hyperelliptic Riemann surface which can be solved explicitly via quadratures. By making use of the gauge transformations of the linear system we were able to show [17] that this is possible in general if the boundary value problem leads to a Riemann-Hilbert problem with rational jump data. Up to now there is however no direct way to infer the jump data from the boundary value problem one wants to solve. The explicit form of the hyperelliptic solutions possibly offers a different approach to boundary value problems: one can try to identify the free parameters in the solutions, a real valued function and a set of complex parameters, the branch points of the hyperelliptic Riemann surface, from the problem one wants to solve.

To this end we study a class of solutions – which is essentially equivalent to [14] and [16] – that is constructed via a generalized Riemann–Hilbert problem on a hyperelliptic Riemann surface. We present a complete discussion of the singularity structure of these Ernst potentials. It is possible to identify a subclass of solutions that are everywhere regular except at some contour, which can possibly be related to the surface of an isolated body, where the Ernst potential is bounded. These solutions are asymptotically flat and equatorially symmetric, and thus show all the features one might expect from the exterior solution for an isolated relativistic ideal fluid. They can have a Minkowskian and an extreme relativistic limit in which the body is 'hidden' behind a horizon, and the exterior solution becomes the extreme Kerr solution. This provides the hope that further solutions to physically interesting boundary value problems to the Ernst equation, besides the rigidly rotating dust disk, can be identified within this class. First results on this subclass where published in [18].

The paper is organized as follows. In section 2 we introduce the linear system associated to the Ernst equation and discuss how the matrix of the system has to be constructed in order to end up with new solutions to the Ernst equation. Using the results of [19], we show how Riemann surfaces arise naturally in the context of linear systems with a spectral parameter. In the case of the Ernst equation, these are hyperelliptic Riemann surfaces with a special structure of the branch points. We will restrict ourselves to regular compact Riemann surfaces and are eventually led to consider families of hyperelliptic Riemann surfaces of arbitrary genus, parametrized by the physical coordinates.

In section 3 we recall some basic notions of the theory of Riemann surfaces, theta functions and the solution of Riemann–Hilbert problems on Riemann surfaces due to Zverovich, and present the class of solutions. It is shown that the solution of the axisymmetric Laplace equation which can be freely prescribed in [16] is a period of the Abelian integrals which determine the singularity structure of the matrix of the linear system. The differential relations between these periods are a subset of the so called Picard–Fuchs equations which we write down for the Ernst equation. In section 4 we discuss the singularity structure of these solutions. It is shown that the solutions can have a regular axis and are in general

asymptotically flat. Using an identity for theta functions, we are able to give in section 5 compact formulas for two metric functions and a simple condition for the occurrence of ergospheres. A subclass of solutions with equatorial symmetry is presented in section 6. The common physical features of this subclass like the extreme relativistic limit are discussed. In section 7, we use the equatorial symmetry to give simplified formulae for the potential in the equatorial plane and on the axis. Since the rigidly rotating dust disk belongs to the simplest non-static solutions which are of genus 2, we consider this case in detail in section 8. In section 9, we summarize the results and add some concluding remarks.

2 Linear System for the Ernst equation and Monodromy matrix

It is well known (see [20]) that the metric of stationary axisymmetric vacuum spacetimes can be written in the Weyl–Lewis–Papapetrou form

$$\mathrm{d}s^2 = -\mathrm{e}^{2U}(\mathrm{d}t + a\mathrm{d}\phi)^2 + \mathrm{e}^{-2U}\left(\mathrm{e}^{2k}(\mathrm{d}\rho^2 + \mathrm{d}\zeta^2) + \rho^2\mathrm{d}\phi^2\right) \tag{2.1}$$

where ρ and ζ are Weyl's canonical coordinates and ∂_t and ∂_ϕ are the two commuting asymptotically timelike respectively spacelike Killing vectors.

In this case the vacuum field equations are equivalent to the Ernst equation for the complex potential f where $f = \mathrm{e}^{2U} + \mathrm{i}b$, and where the real function b is related to the metric functions via

$$b_{,z} = -\frac{\mathrm{i}}{\rho}\mathrm{e}^{4U}a_{,z}. \tag{2.2}$$

Here the complex variable z stands for $z = \rho + \mathrm{i}\zeta$. With these settings, the Ernst equation reads

$$f_{z\bar{z}} + \frac{1}{2(z+\bar{z})}(f_{\bar{z}} + f_z) = \frac{2}{f+\bar{f}}f_z f_{\bar{z}}\ , \tag{2.3}$$

where a bar denotes complex conjugation in $\mathbb{C}$. With a solution f, the metric function U follows directly from the definition of the Ernst potential whereas a can be obtained from (2.2) via quadratures. The metric function k can be calculated from the relation

$$k_{,z} = 2\rho\,(U_{,z})^2 - \frac{1}{2\rho}\mathrm{e}^{4U}\,(a_{,z})^2\,. \tag{2.4}$$

The integrability condition of (2.2) and (2.4) is the Ernst equation.

The remarkable feature of the Ernst equation is that it is completely integrable. This means that it can be considered as the integrability condition of an overdetermined linear differential system for a matrix valued function Φ that contains an additional variable, the so called spectral parameter K. The occurrence of the linear system with a spectral parameter is a consequence of the symmetry group of the Ernst equation, the Geroch group [8]. Several forms of the linear system are known in the literature ([9, 10, 11]). They are related through gauge transformations (see [21]). The choice of a specific form of the linear system

is equivalent to a gauge fixing. We will use the form of [11],

$$\Phi_{,z}(K,\mu_0;z,\bar z) = \left\{\begin{pmatrix} N & 0 \\ 0 & M\end{pmatrix} + \frac{K-\mathrm{i}\bar z}{\mu_0(K)}\begin{pmatrix} 0 & N \\ M & 0\end{pmatrix}\right\}\Phi(K,\mu_0;z,\bar z) \doteq W\Phi \ , \tag{2.5 a}$$

$$\Phi_{,\bar z}(K,\mu_0;z,\bar z) = \left\{\begin{pmatrix} \bar M & 0 \\ 0 & \bar N\end{pmatrix} + \frac{K+\mathrm{i} z}{\mu_0(K)}\begin{pmatrix} 0 & \bar M \\ \bar N & 0\end{pmatrix}\right\}\Phi(K,\mu_0;z,\bar z) \doteq V\Phi \tag{2.5 b}$$

where

$$M = \frac{f_z}{f+\bar f}, \quad N = \frac{\bar f_z}{f+\bar f}. \tag{2.6}$$

Obviously M and N depend only on the coordinates z and $\bar z$ and not on the spectral parameter K that lives on the Riemann surface $\mathcal{L}(z,\bar z) = \mathcal{L}$ given by $\mu_0^2(K) = (K-\mathrm{i}\bar z)(K+\mathrm{i}z)$. Notice that $\mathcal{L}$ is a Riemann surface of genus zero with coordinate dependent branch points. This is a special feature of the family of chiral field equations to which the Ernst equation belongs that has no counterpart among the completely integrable nonlinear evolution equations for which algebro-geometric solutions have been constructed first.

On $\mathcal{L}$ we have an involutive map σ, defined by

$$\mathcal{L} \ni P = (K, \pm\sqrt{(K-\mathrm{i}\bar z)(K+\mathrm{i}z)}) \to \sigma(P) \equiv P^\sigma = (K, \mp\sqrt{(K-\mathrm{i}\bar z)(K+\mathrm{i}z)}) \in \mathcal{L} \ , \tag{2.7}$$

and an anti–holomorphic involution τ, defined by

$$\mathcal{L} \ni P = (K, \pm\sqrt{(K-\mathrm{i}\bar z)(K+\mathrm{i}z)}) \to \tau(P) \equiv \bar P = (\bar K, \pm\sqrt{(\bar K-\mathrm{i}\bar z)(\bar K+\mathrm{i}z)}) \in \mathcal{L} \ . \tag{2.8}$$

It is possible to use the existence of the above linear system for the construction of solutions to the Ernst equation. To this end one investigates the singularity structure of the matrices $\Phi_z\Phi^{-1}$ and $\Phi_{\bar z}\Phi^{-1}$ with respect to the spectral parameter and infers a set of conditions for the matrix Φ (at least twice differentiable with respect to z, $\bar z$) that satisfies the linear system (2.5 a) and (2.5 b). This is done (see e.g. [14]) in

Theorem 2.1 *Let $\Phi(P)$ ($P \in \mathcal{L}$) be a 2×2–matrix with the following properties:*
I. $\Phi(P)$ is holomorphic and invertible at the branch points $P_0 = -\mathrm{i}z$ and $\bar P_0$ such that the logarithmic derivative $\Phi_z\Phi^{-1}$ diverges as $(K+\mathrm{i}z)^{\frac12}$ at P_0 and $\Phi_{\bar z}\Phi^{-1}$ as $(K-\mathrm{i}\bar z)^{\frac12}$ at $\bar P_0$.
II. All singularities of Φ on $\mathcal{L}$ (poles, essential singularities, zeros of the determinant of Φ, branch cuts and branch points) are regular which means that the logarithmic derivatives $\Phi_z\Phi^{-1}$ and $\Phi_{\bar z}\Phi^{-1}$ are holomorphic in the neigbourhood of the singular points (this implies they have to be independent of z, $\bar z$). In particular $\Phi(P)$ should have
a) regular singularities at the points $A_i \in \mathcal{L}$ ($i = 1,\ldots,n$) which do not depend on z, $\bar z$,
b) regular essential singularities at the points S_i ($i = 1,\ldots,m$) which do not depend on z, $\bar z$,
c) boundary values at a set of (orientable, piecewise smooth) contours $\Gamma_i \subset \mathcal{L}$ ($i = 1,\ldots,l$) independent of z, $\bar z$, which are related on both sides of the contours via

$$\Phi_-(P) = \Phi_+(P)\mathcal{G}_i(P)|_{P\in\Gamma_i} \ . \tag{2.9}$$

where $\mathcal{G}_i(P)$ are matrices independent of z, $\bar z$ with Hölder–continuous components and non–vanishing determinant.
III. Φ satisfies the reduction condition

$$\Phi(P^\sigma) = \sigma_3\Phi(P)\gamma \ , \tag{2.10}$$

where σ_3 is the third Pauli matrix, and where γ is an invertible matrix independent of z and $\bar{z}$.
IV. The normalization and reality condition

$$\Phi(P = \infty^+) = \begin{pmatrix} \bar{f} & 1 \\ f & -1 \end{pmatrix}. \tag{2.11}$$

Then the function f in (2.11) is a solution to the Ernst equation.

A proof of this Theorem may be obtained by comparing the above matrix Φ with the linear system (2.5 a) and (2.5 b).

Proof : Because of I, Φ and Φ^{-1} can be expanded in a series in $t = \sqrt{K + iz}$ and $t' = \sqrt{K - i\bar{z}}$ in a neighbourhood of $P = P_0$ and $P = \bar{P}_0 \neq P_0$ respectively at all points P_0, $\bar{P}_0$ which do not belong to the singularities given in II. This implies that $\Phi_z \Phi^{-1} = \alpha_0/t + \alpha_1 + \alpha_2 t + \cdots$. We recognize that, because of I and II, $\Phi_z \Phi^{-1} - \alpha_0/t$ is a holomorphic function. The normalization condition IV implies that this quantity is bounded at infinity. According to Liouvilles theorem, it is a constant. Since Φ, Φ^{-1} and Φ_z are single valued functions on $\mathcal{L}$, they must be functions of K and μ_0. Therefore we have $\Phi_z \Phi^{-1} = \beta_0 \sqrt{\frac{K - \bar{P}_0}{K - P_0}} + \beta_1$. The matrix β_0 must be independent of K and μ since $\Phi_z \Phi^{-1}$ must have the same number of zeros and poles on $\mathcal{L}$. The structure of the matrices β_0 and β_1 follows from III. From the normalization condition IV, it follows that $\Phi_z \Phi^{-1}$ has the structure of (2.5 a). The corresponding equation for $\Phi_{\bar{z}} \Phi^{-1}$ can be obtained in the same way. □

For a given Ernst potential f, the matrix Φ in the above theorem is not uniquely determined. This reflects the fact that the gauge is not uniquely fixed in the linear system (2.5 a) and (2.5 b). If we choose without loss of generality $\gamma = \sigma_1$ (the first Pauli matrix), the remaining gauge freedom can be seen from

Corollary 2.2 *Let $\Phi(P)$ be a matrix subject to the conditions of Theorem 2.1, and $C(K)$ be a 2×2-matrix that only depends on $K \in \mathbb{C}$ with the properties*

$$\begin{aligned} C(K) &= \alpha_1(K)\hat{1} + \alpha_2(K)\sigma_1, \\ \alpha_1(\infty) &= 1, \quad \alpha_2(\infty) = 0. \end{aligned} \tag{2.12}$$

Then the matrix $\Phi'(P) = \Phi(P)C(K)$ also satisfies the conditions of Theorem 2.1 and $\Phi'(\infty^+) = \Phi(\infty^+)$.

It is this gauge freedom to which we refer when we speak of the gauge freedom of the linear system in the following.

It is interesting to note that the metric function a can be obtained from a given matrix Φ without solving the equation (2.2), see [14]. We get

Proposition 2.3 *Let δ be a local parameter in the vicinity of ∞^-. Then*

$$(a - a_0)e^{2U} = \mathrm{i}(\Phi_{11} - \Phi_{12})_{,\delta} \ , \tag{2.13}$$

where a_0 is a constant that is fixed by the condition that $a = 0$ on the regular part of the axis and at spatial infinity, and where $\Phi_{,\delta}$ denotes the linear term in the expansion of Φ in δ divided by δ.

The proof follows from the linear system (2.5 a) and (2.5 b).

Proof : It is straightforward to check the relation

$$(\Phi^{-1}\Phi_{,\delta})_{,z} = \Phi^{-1}(\Phi_z\Phi^{-1})_{,\delta}\Phi \ . \tag{2.14}$$

With (2.5 a), we get

$$\left(\Phi^{-1}\Phi_\delta\right)_{21,z} = \frac{\mathrm{i}\rho}{(f+\bar{f})^2}(\bar{f}-f)_z \ , \tag{2.15}$$

from which, together with (2.2), (2.13) follows. □

Notice that a_0 is not gauge independent (in the sense of the above corollary) whereas a is.

Theorem 2.1 can be used to construct solutions to the Ernst equation by determining the structure and the singularities of Φ in accordance with the conditions I–IV. For nonlinear evolution equations, large classes of solutions were constructed with the help of algebro-geometric methods, in particular Riemann surface techniques. A keypoint in this context is the occurrence of Riemann surfaces which are related to the linear system of the integrable equation under consideration. In this paper we want to show how solutions for the Ernst equation can be constructed by making use of the so called monodromy matrix of the Ernst system, which – following [19] – can be introduced as follows.

For a given linear system (2.5 a) and (2.5 b), we define the monodromy matrix L as a solution to the system

$$L_z = [W, L], \quad L_{\bar{z}} = [V, L] \ . \tag{2.16}$$

For a known solution Φ of (2.5 a) and (2.5 b), L can be directly constructed in the form

$$L(K) = -\hat{\mu}(K)\Phi\mathcal{C}\Phi^{-1} \tag{2.17}$$

where $\mathcal{C}$ is an arbitrary constant matrix with $\det\mathcal{C} = -1$ and $\hat{\mu}$ does not depend on the physical coordinates. Since Φ is analytic in K, there is a solution to (2.16) with the same properties.

It follows from (2.16) that the coefficients of the characteristic polynomial $Q(\mu, K) = \det(L(K) - \hat{\mu}\hat{1})$ are independent of the coordinates. Without loss of generality we may assume $\mathrm{Tr}L(K) = 0$. Then L has the structure

$$L = \begin{pmatrix} A(K) & B(K) \\ C(K) & -A(K) \end{pmatrix} \ . \tag{2.18}$$

The equation $Q(\hat{\mu}, K) = 0$, i.e.

$$\hat{\mu}^2 = A^2 + BC \ , \tag{2.19}$$

is then the equation of an algebraic curve which in general will have infinite genus. We will restrict the analysis in the following to the case of a regular curve with finite genus.

In this case, the Riemann surface $\hat{\mathcal{L}}$ is given by an equation of the form

$$\hat{\mu}^2 = \prod_{i=1}^{g}(K - E_i)(K - F_i) \tag{2.20}$$

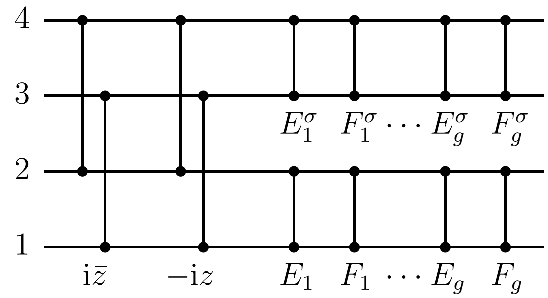

Figure 1: The Hurwitz diagram of $\hat{\mathcal{L}}$.

where E_i and F_i are obviously independent of the physical coordinates. This equation represents a two sheeted covering of the Riemann sphere and thus a four sheeted covering of the complex plane. A point $\hat{P} \in \hat{\mathcal{L}}$ can be given by $\hat{P} = (K, \mu_0(K), \hat{\mu}(K))$. The Hurwitz diagram of $\hat{\mathcal{L}}$ is shown in figure 1.

There is an automorphism σ of $\hat{\mathcal{L}}$ inherited from $\mathcal{L}$ which ensures $E_i^\sigma = E_i$ and $F_i^\sigma = F_i$. The orbit space $\mathcal{L}_H = \hat{\mathcal{L}}/\sigma$ is then, see [13], again a Riemann surface, namely a hyperelliptic surface given by

$$\mu_H^2 = (K - \mathrm{i}\bar{z})(K + \mathrm{i}z) \prod_{i=1}^{g} (K - E_i)(K - F_i) \ . \tag{2.21}$$

Thus it is possible to construct components of the matrix Φ on $\mathcal{L}_H$ which makes it possible to use the powerful calculus of hyperelliptic Riemann surfaces. These functions may be lifted to $\hat{\mathcal{L}}$. As we will show in the following, it is possible to construct a matrix Φ on $\mathcal{L}$ in accordance with the conditions of Theorem 2.1 by projecting onto this surface.

3 Hyperelliptic solutions of the Ernst equation

3.1 Theta functions asscociated with a Riemann surface and the Riemann–Hilbert problem

In this section, we want to give an explicit construction of the matrix Φ in accordance with Theorem 2.1. Condition II can be used to construct solutions by prescribing the poles, essential singularities and cuts of Φ which is equivalent to the solution of a generalized Riemann–Hilbert problem for the matrix Φ. The investigation of such matrix Riemann–Hilbert problem turns out to be rather difficult and is not yet fully done (in general it can be merely reduced to the solution of a linear integral equation, see e.g. [22]). Therefore we will use here a different approach. The occurrence of the monodromy matrix suggests that it might be possible to construct a matrix Φ on the Riemann surface $\hat{\mathcal{L}}$ of the previous section. The additional freedom we thus gain is used to restrict the problem to a scalar one, namely to a Riemann–Hilbert problem for one component of Φ on the hyperelliptic surface $\mathcal{L}_H$ obtained from $\hat{\mathcal{L}}$ by factorizing with respect to the involution σ. We impose the reality condition $E_i, F_i \in \mathbb{R}$ or $E_i = \bar{F}_i$ on the branch points in order to satisfy the reality condition

of Theorem 2.1. Then we construct the whole matrix Φ in accordance with this theorem. In fact it was shown in [17] that all matrix Riemann–Hilbert problems with rational jump data are gauge equivalent to scalar problems on a suitably chosen hyperelliptic surface. Thus the limitation to the scalar case is only a comparatively weak restriction which allows, as we will show below, for an explicit solution of the problem in terms of theta functions.

For the moment, we fix the physical coordinates z and $\bar{z}$ in a way that $\rho \neq 0$ and that $-\mathrm{i}z$ and $\mathrm{i}\bar{z}$ do not coincide with the singular points of Φ in order to ensure that the first condition of Theorem 2.1 is valid. In the next section we study the dependence of the found solution on z and $\bar{z}$. In order to give the solution to this special case of the generalized Riemann–Hilbert problem, we use the theory of theta functions associated to a Riemann surface (see [23]) and the solution of the Riemann–Hilbert problem on a Riemann surface, as given in [25]. As we will need only hyperelliptic Riemann surfaces of the form (2.21), we restrict ourselves to this case.

Let us denote by $(a_1, \ldots, a_g, b_1, \ldots, b_g)$ a basis of the first (integral) homology group $H_1(\mathcal{L}_H)$ of $\mathcal{L}_H$ (see the picture below) where the cuts are either between real branch points (which are ordered $E_{k+1} < F_{k+1} < \ldots$) or between E_i and $\bar{E}_i$ (for the moment we ignore the case that more than two branch points may have the same real part).

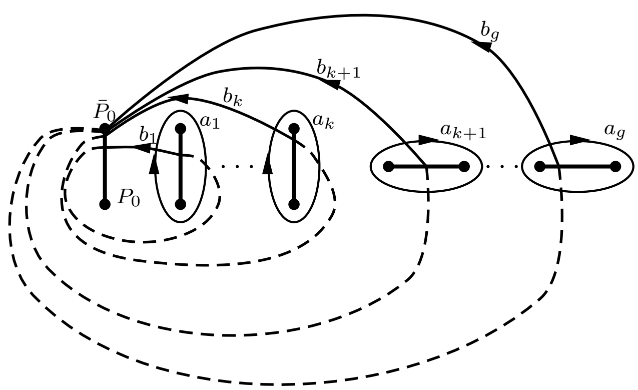

Figure 2: The homology basis for $\mathcal{L}_H$.

Let $\{\mathrm{d}\omega_i\}$ denote a basis of $H^1(\mathcal{L}_H)$ such that $\oint_{a_i} \mathrm{d}\omega_j = 2\pi\mathrm{i}\delta_{ij}$. Using these normalized differentials, we define the Abel–Jacobi map of $P_H \in \mathcal{L}_H$ by $\omega(P_H) = (\int_{P_0}^{P_H} \mathrm{d}\omega_1, \ldots, \int_{P_0}^{P_H} \mathrm{d}\omega_g)$ with $P_0 \in \mathcal{L}_H$ fixed. In the following, we will always choose $P_0 = -\mathrm{i}z$.

We define a $g \times g$ matrix Π – the Riemann matrix – with elements $\pi_{ij} \doteq \oint\limits_{b_i} \mathrm{d}\omega_j$. This matrix is symmetric and has a negative definite real part. These properties ensure that the theta function with integer characteristic $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ defined by

$$\Theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (x, \Pi) = \sum_{N \in Z^g} \exp \left\{ \frac{1}{2} \left\langle \Pi \left(N + \frac{\alpha}{2} \right), N + \frac{\alpha}{2} \right\rangle + \left\langle x + \pi \mathrm{i} \beta, N + \frac{\alpha}{2} \right\rangle \right\} \,, \tag{3.1}$$

with $x \in \mathbb{C}^g$ and α, $\beta \in \mathbb{Z}^g$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product $\langle N, x\rangle = \sum_{i=1}^{g} N_i x_i$, is an analytic function on $\mathbb{C}^g$. A characteristic is called odd if $\langle \alpha, \beta\rangle \neq 0 \mod 2$. The Riemann vector is denoted by K_R.

The reality condition on the branch points implies for the theta function Θ with characteristic $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, the Riemann theta function,

$$\bar{\Theta}(x) = \Theta(\bar{x} + \mathrm{i}\pi\Delta) \ , \tag{3.2}$$

where $\Delta_i = 1$ if E_i and F_i are real and $\Delta_i = 0$ otherwise.

We recall that a divisor $\mathfrak{A}$ on a general Riemann surface Σ_g is a formal symbol $\mathfrak{A} = n_1 P_1 + \cdots n_k P_k$ with $P_i \in \Sigma_g$ and $n_i \in \mathbb{Z}$. The set of divisors admits a partial ordering and we may associate to a meromorphic function f a divisor (f), a principal divisor. Using this partial ordering we define for a divisor $\mathfrak{A}$ a vector space $L(\mathfrak{A})$ consisting of all principal divisors not less than $\mathfrak{A}$.

A Riemann–Hilbert problem can be stated as follows: let Γ be a piecewise smooth contour on $\mathcal{L}_H$. Let $\Lambda = t_1 + \cdots + t_r$ be a divisor on Γ consisting of a finite number of pairwise different points subject to the following condition: $\Gamma \setminus \Lambda$ decomposes into a finite set of connected components $\{\Gamma_j\}$ $(j = 1, \ldots, N)$, each of which is homeomorphic to the interval $(0, 1)$. We call Γ_j a *curve of the contour* Γ. Each Γ_j has a starting and an end point, given by two points of Λ, where the starting respectively end points may also coincide. We define the function $\alpha(t, \Gamma_j)$ on Γ by

$$\alpha(t, \Gamma_j) = \begin{cases} 1 \text{ if } t \in \Gamma_j \\ 0 \text{ otherwise} \end{cases} , \tag{3.3}$$

$(j = 1, \ldots, N)$. On each curve Γ_j let there be defined a Hölder-continuous function $G_j(t)$, which is finite and nonzero. We denote

$$G(t) = \sum_{j=1}^{N} \alpha(t, \Gamma_j) G_j(t) \ , \quad t \in \Gamma \setminus \Lambda. \tag{3.4}$$

Let there be given a divisor $\mathfrak{A}$ of degree m, consisting of points of the divisor Λ, taken at arbitrary degree. Let on $\mathcal{L}_H \setminus \Gamma$ be given another divisor $\mathfrak{B}$ of degree n. Now we can formulate the

Homogeneous scalar Riemann–Hilbert problem:
Give a function ψ with the properties

$$\psi^+(t) = G(t)\psi^-(t) \ , \tag{3.5}$$

with $(\psi) \in L(\mathfrak{A}^{-1}\mathfrak{B}^{-1})$.

The solution of this problem was given by Zverovich [25]. A key point in the construction is the introduction of an analogue to the usual Cauchy kernel. This Cauchy analogue is given by a normalized (i.e. all a-periods are zero) differential of the third kind with poles at P_H and P_0 and is denoted by $\mathrm{d}\omega_{P_H P_0}(\tau)$. With the Cauchy analogue at hand the solution to the above Riemann-Hilbert problem is given by

$$\psi(P_H) = \mathrm{e}^{\Psi(P_H)} \ , \tag{3.6}$$

with

$$\Psi(P_H) = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \ln G(\tau)\, \mathrm{d}\omega_{P_H P_0}(\tau) \tag{3.7}$$

where $P_0 \notin \Gamma$, and the integration goes over all curves Γ_j of the contour Γ, and where we have put $\ln G(t) = \sum_{j=1}^{N} \alpha(t, \Gamma_j) \ln G_j(t)$. The b-periods u_i of ψ are given by

$$u_i = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \ln G\, \mathrm{d}\omega_i \quad . \tag{3.8}$$

Applying the Plemelj formulae to the function $\Psi(P_H)$, we find

$$\psi^+(t) = G(t)\psi^-(t) \ , \quad t \in \Gamma \setminus \Lambda \setminus \bigcup_{i=1}^{g} a_i. \tag{3.9}$$

For more details on the solution to the scalar Riemann–Hilbert problem, the reader is referred to [25], [26] or [27].

3.2 Solutions to the Ernst equation via the scalar Riemann–Hilbert problem

With the help of the results of the previous section, we are now able to state the following theorem, which gives the solution to the generalized Riemann–Hilbert problem on the hyperelliptic Riemann surface $\mathcal{L}_H$.

Theorem 3.1 *Let P_0 be a fixed complex constant ($\rho \neq 0$) not coinciding with the singularities of ψ or the branch points E_i or F_i. Let $\Omega(P_H)$ be a linear combination of normalized Abelian integrals of the second kind (with singularities $p \neq E_i$ and $p \neq F_i$, independent of z and $\bar{z}$) and third kind (with in addition singularities at all real branch points with residues $\pm\frac{1}{2}$), satisfying $\bar{\Omega}(P_H) = \Omega(\bar{P}_H)$. Then the solution to the generalized Riemann–Hilbert problem on the real hyperelliptic Riemann surface $\mathcal{L}_H$ is given by*

$$\psi(P_H) = \psi_0 \frac{\Theta(\omega(P_H) - \omega(D) + u + b - K_R)}{\Theta(\omega(P_H) - \omega(D) - K_R)} \exp\left\{ \Omega(P_H) + \frac{1}{2\pi\mathrm{i}} \int_\Gamma \ln G(\tau) \mathrm{d}\omega_{P_H P_0}(\tau) \right\} \ , \tag{3.10}$$

where $D = P_1 + \cdots + P_g$ is a fixed non–special divisor on $\mathcal{L}_H$ which is subject to the reality condition: either $P_i \in \mathbb{R}$ or with $P_i \in D$ we have $\bar{P}_i \in D$ or P_i is a branch point E_i or F_i. b is the vector of b-periods of Ω with components

$$b_i = \oint_{b_i} \Omega \ , \tag{3.11}$$

$i = 1, \ldots, g$, the u_i are given by (3.8), and ψ_0 is a normalization constant. The paths of integration have to be the same for all integrals.

Proof : We want to prove that $\psi(P_H)$ is a single valued function on $\mathcal{L}_H$. If we choose a different path of integration for the integrals in the exponent and the map $\omega(P_H)$ and denote the corresponding integrals by a prime, the primed and unprimed integrals are connected via

$$\Omega'(P_H) = \Omega(P_H) + \oint_{\mathcal{E}} \mathrm{d}\Omega \ , \tag{3.12}$$

(similarly for the other integrals) where $\mathcal{E}$ is a closed contour on $\mathcal{L}_H$ which may be decomposed in the homology basis as follows

$$\mathcal{E} = \sum_{i=1}^{g} m_i \, a_i + \sum_{i=1}^{g} n_i \, b_i \ , \tag{3.13}$$

with m_i, $n_i \in \mathbb{Z}$. Then we have, e.g. for Ω and ω

$$\begin{aligned} \Omega(P_H) &\to \Omega(P_H) + \sum_{i=1}^{g} n_i b_i = \Omega(P_H) + \langle N, b \rangle \ , \\ \omega(P_H) &\to \omega(P_H) + 2\pi \mathrm{i} M + \mathit{\Pi} N \ , \end{aligned} \tag{3.14}$$

where $M = (m_1, \ldots, m_g)$, $N = (n_1, \ldots, n_g) \in \mathbb{Z}^g$. Under this transformation, the original quotient of theta functions in (3.10) will be multiplied by

$$\exp\left(- \langle N, b \rangle\right) \ , \tag{3.15}$$

but this term is just compensated by the contour integral over $\mathcal{E}$ in the exponent. The same argument holds for the line integral over the contour Γ since the u_i are its b-periods. This shows that $\psi(P_H)$ is a single valued function on $\mathcal{L}_H$.

¿From the properties of the theta function, we also find that $\psi(P)$ has g simple poles at the points $P_1, \ldots, P_g$ and g simple zeros. Additional poles, zeros and essential singularities can be obtained by a suitable choice of Abelian integrals of the second kind (essential singularities) and third kind (zeros and poles). We remark that the assumption $\bar{\Omega}(P) = \Omega(\bar{P})$ had to be introduced in order to satisfy the reality condition of Theorem 2.1. □

Remark : Without loss of generality we can choose D to consist only of branch points since D gives the poles of Ψ due to the zeros of the theta function in the denominator. This can always be compensated by a suitable choice of the zeros and poles of Ψ which arise from the integrals of the third kind in Ω. All $P_i \in D$ shall have multiplicity 1 and be chosen in a way that $\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(x)$ with $\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \omega(D) + \omega(\bar{P}_0) + K_R$ has the same reality properties as the Riemann theta function $\Theta(x)$.

Our next aim is to define a matrix valued function $\Phi(P)$ on $\mathcal{L}$, satisfying the conditions of theorem 2.1, with the help of the above solution to the scalar Riemann–Hilbert problem on the hyperelliptic surface $\mathcal{L}_H$. To this end we define a further function on $\mathcal{L}_H$ by

$$\chi(P_H) = \chi_0 \frac{\Theta(\omega(P_H) + u - \omega(\bar{P}_0) - \omega(D) - K_R)}{\Theta(\omega(P_H) - \omega(D) - K_R)} \exp\left(\frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \ln G \mathrm{d}\omega_{P_H P_0}\right), \tag{3.16}$$

where χ_0 is again a normalization constant. It can be easily seen that the analytic behaviour of $\chi(P_H)$ is identical to that of $\psi(P_H)$, except that it changes the sign at every a-cut. χ is thus not a single valued function on $\mathcal{L}_H$. However, it is single valued on $\hat{\mathcal{L}}$ which can be viewed as two copies of $\mathcal{L}_H$ cut along $\left[P_0, \bar{P}_0\right]$ and glued together along this cut. We define the vector X on $\hat{\mathcal{L}}$ by fixing the sign in front of χ in the vicinity of the points $P_0^{\pm} = (K_0, 0, \pm\hat{\mu}(K_0)) \in \hat{\mathcal{L}}$,

$$X(\hat{P}) = \begin{pmatrix} \psi(\hat{P}) \\ \pm\chi(\hat{P}) \end{pmatrix}, \quad \hat{P} \sim P_0^{\pm}. \tag{3.17}$$

With the help of this vector, we can construct the matrix Φ on $\mathcal{L}$ via

$$\Phi(P) = (X(K, \mu_0(K), +\hat{\mu}(K)), X(K, \mu_0(K), -\hat{\mu}(K))) \tag{3.18}$$

where the signs are again fixed in the vicinity of $P_0^{\pm}$. Notice that this matrix consists of eigenvectors of the monodromy matrix, $LX(K, \mu_0(K), \pm\hat{\mu}(K)) = \hat{\mu}X(K, \mu_0(K), \pm\hat{\mu}(K))$ if L is written as $L = \hat{\mu}\Phi\gamma\Phi^{-1}$ where γ is the matrix from 2.1.

It may be readily checked that this ansatz is in accordance with the reduction condition (2.10) (this is in fact the reason why one has to define the function χ in the way (3.16)). The behaviour at the singularities is as required in condition II: For the contour Γ and the singularities of the Abelian integrals Ω, this is obvious. At the branch points E_i and F_i, one gets the following behaviour: at points P_i of the divisor D, the components of Φ have a simple pole, and the determinant diverges as $(K - P_i)^{-\frac{1}{2}}$, if this branch point is not a singularity of an integral of the third kind in Ω or lies on the contour Γ. If the same condition holds at the remaining branch points, the components are regular there but the determinant vanishes as $(K - P_i)^{\frac{1}{2}}$. If the branch points coincide with one of the singularities of the integrals in the exponent in (3.10), this merely changes the singular behaviour of Φ and its determinant there. Condition II of theorem 2.1 is however obviously satisfied.

Since Φ in (3.18) is only a function of P, it will not be regular at the cuts $[E_i, F_i]$. At the a-cuts around non-real branch points, we get $\Phi^- = \Phi^+\sigma_1|_{a_i}$, whereas we have $\Phi^-|_{a_i} = -\Phi^+|_{a_i}\sigma_2$ at the a-cuts around real branch points. The logarithmic derivatives of Φ with respect to z and $\bar{z}$ are however holomorphic at all these points. One can recognize that the behaviour at the non-real branch points is related to a gauge transformation of the form (2.12). This means that one can find a gauge transformed matrix Φ' that is completely regular at these points if the integrals in the exponent are regular there. With

$$\alpha_1 = \frac{1}{2}(1 + \lambda), \quad \alpha_2 = \frac{1}{2}(1 - \lambda) \tag{3.19}$$

and $\lambda = \prod_{i=1}^{g} \sqrt{\frac{K - \bar{P}_i}{K - P_i}}$ where $D = \sum_{i=1}^{g} P_i$, this may be checked by direct calculation. The real branch points, however, cannot be related to gauge transformations.

Normalizing ψ and χ (if possible) in a way that $\psi(\infty_H^-) = 1$ and $\chi(\infty_H^-) = -1$, one can see that Φ is then in accordance with all conditions of Theorem 1 since the reality condition follows from the reality properties of the theta functions and the Riemann–Hilbert problem. The fact that Φ is at least differentiable with respect to z and $\bar{z}$ at points where P_0 does not coincide with the singularities of the integrals in the exponent or the remaining branch points of $\mathcal{L}_H$ follows from the modular properties of the theta function. Let the

paths between $[P_0, \infty^-]$ and $[P_0, \infty^+]$ be the same in all integrals and let them have the same projection into the complex plane (i.e. one is the involuted of the other). Then the results may be summarized in

Theorem 3.2 *Let* $\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\omega(\infty^-) + u) \neq 0$. *Then the function*

$$f(z, \bar{z}) = \frac{\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\omega(\infty^+) + u + b)}{\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\omega(\infty^-) + u + b)} \exp\left\{\Omega(\infty^+) - \Omega(\infty^-) + \frac{1}{2\pi \mathrm{i}}\int_\Gamma \ln G(\tau) \mathrm{d}\omega_{\infty^+\infty^-}(\tau)\right\} \tag{3.20}$$

is a solution to the Ernst equation.

Remarks :

1. In the case $g = 0$ the Ernst potential (3.20) is real, $f = \mathrm{e}^{2U}$. This means that U is a solution to the axisymmetric Laplace equation and belongs therefore to the Weyl-class. For $g > 0$, there are no real solutions other than $f = 1$ which describes Minkowski space.

2. The multi-black-hole solutions which can be obtained via Bäcklund transformations (see e.g. [28]) are contained in the class (3.20) as the limiting case that the branch points E_i and F_i coincide pairwise. In this limit, all branch points become double points and the theta functions break down to purely algebraic functions. Notice that the analysis of f at the branch points in the following section always assumes a regular surface. The obtained results for the regularity of f do not hold in this limit.

The above explicit construction of the solutions makes it possible to derive useful formulae for the metric function a and the derivatives of the Ernst potential. Let $\int_{P_H}^{P_H+\delta} \mathrm{d}\omega_i = g_i\delta + o(\delta)$ where δ is the local parameter in the vicinity of $P_H \in \mathcal{L}_H$. We define the derivative

$$D_{P_H}\Theta(x) = \sum_{i=1}^{g} g_i \partial_{x_i}\Theta(x). \tag{3.21}$$

Using (2.13) and (3.10), (3.16), we get

$$(a - a_0)\mathrm{e}^{2U} = \mathrm{i}D_{\infty^-} \ln \frac{\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]\left(\int_{\bar{P}_0}^{\infty^-} d\omega + u + b\right)}{\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]\left(\int_{P_0}^{\infty^-} d\omega + u + b\right)}. \tag{3.22}$$

¿From the linear system (2.5 a) and (2.5 b), we obtain with (3.10) and (3.16)

$$\frac{\bar{f}_z}{f + \bar{f}} = \frac{\mathrm{i}}{2\sqrt{P_0 - \bar{P}_0}} \frac{\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](u + b - \omega(\bar{P}_0))\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](u + b + \omega(\infty^-))}{\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](u + b)\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](u + b + \omega(\infty^-) - \omega(\bar{P}_0))} \times$$
$$\left(D_{P_0} \ln \Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(u - \omega(\bar{P}_0)) + I_{P_0}\right) \tag{3.23}$$

and

$$\frac{f_z}{f+\bar{f}} = \frac{\mathrm{i}}{2\sqrt{P_0-\bar{P}_0}} \frac{\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](u+b)\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](u+b+\omega(\infty^-)-\omega(\bar{P}_0))}{\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](u+b-\omega(\bar{P}_0))\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](u+b+\omega(\infty^-))} \times$$
$$\left(D_{P_0}\ln\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(u-\omega(\bar{P}_0))+I_{P_0}\right), \tag{3.24}$$

where I_{P_0} is the linear term of the expansion of the integrals in the exponent of (3.10) in the local parameter around P_0.

3.3 Finite gap solutions and Picard–Fuchs equations

The original finite gap solutions of [14] are those among (3.20) without the contour integral (in our notation only an arbitrary linear combination of Abelian integrals of the second and third kind Ω). They just correspond to the so called Baker–Akhiezer function (see [13]) for the Ernst system. This function that has essential singularities and poles gives the periodic or quasiperiodic solutions to the integrable nonlinear evolution equations. There the essential singularity is uniquely determined by the structure of the differential equation. In contrast to these equations, the solutions (3.20) are in general neither periodic nor quasiperiodic, and the essential singularity can be nearly arbitrarily chosen. The form of the solution to the Riemann–Hilbert problem shows that one might even think of "putting the singularities densely on a line and integrate over the integrals with some measure": an Abelian integral Ω_p of the second kind with a pole of first order at p can be used as an analogue to the Cauchy kernel. A contour integral over this kernel with some measure, $\int_\Gamma \ln G\Omega_p \mathrm{d}p$, is thus just another way to write down the solution to a Riemann–Hilbert problem on a Riemann surface.

In studying the boundary value problem for the rigidly rotating disk of dust, Meinel and Neugebauer [16] observed that it is possible to obtain solutions to the Ernst equations via

$$f = \exp\left(\sum_{m=1}^{g}\int_{E_m}^{C_m}\frac{K^g\mathrm{d}K}{\mu_H} - I_g\right) \tag{3.25}$$

where the divisor $C=\sum_{m=1}^{g}C_m$ is determined by

$$\sum_{m=1}^{g}\int_{E_m}^{C_m}\frac{K^i\mathrm{d}K}{\mu_H} = I_i \tag{3.26}$$

($i=0,1,\ldots,g-1$), i.e. as the solution of a Jacobi inversion problem. The I_i are (in the absence of real branch points) real solutions to the axisymmetric Laplace equation which satisfy the recursive condition,

$$\mathrm{i}I_{n+1,z} = zI_{n,z}+\frac{1}{2}I_n \ . \tag{3.27}$$

The relation to the class obtained in theorem 3.2 is the following: The integral of the third kind in (3.25) can be expressed by the help of a formula in [33] via theta functions.

Equation (3.26) ensures that the resulting expression is independent of the chosen integration path which is shown in the proof of the theorem. Thus the u_i (obtained from the I_i by normalization) are as in our case the b-periods of the integral I_g in the exponent. In fact, it was shown in [16] that one of these periods, say I_1, can be chosen as an arbitrary solution to the axisymmetric Laplace equation. The other periods as well as the integral in the exponent then follow from differential identities plus boundary conditions.

The underlying reason for this fact is that the Ernst potential f is studied on a family of Riemann surfaces parametrized by the moving branch points $-\mathrm{i}z$ and $\mathrm{i}\bar{z}$. The periods on this surface (i.e. integrals along closed curves) are subject to differential identities, the so called Picard–Fuchs equations. It is a general feature of the periods of rational functions [29, 30, 31] that they satisfy a differential system of finite order with Fuchsian singularities. An elegant way to find the Picard–Fuchs system explicitly is via the notion of the Manin connection in the bundle $H^1_{\mathrm{DR}}(\Sigma_g) \to \Sigma_g$, see [32]. The investigation turns out to be particularly simple if one uses the following standard form of the (hyperelliptic) Riemann surface Σ_g (all hyperelliptic surfaces of genus g are conformally equivalent to this standard form)

$$y^2 = (x-z)\prod_{i=1}^{2g}(x-E_i) \doteq (x-z)P(x) = (x-z)\sum_{j=0}^{2g} a_j x^j \ , \tag{3.28}$$

where the E_i do not depend on z. Using $j_0 = \mathrm{d}x/y$, $j_1 = xj_0, \ldots, j_{2g-1} = x^{2g-1}j_0$ as the basis for the de Rham cohomology $H^1_{\mathrm{DR}}(\Sigma_g)$ we obtain for the matrix M_n^m $(m, n = 0, \ldots, 2g-1)$ of the Manin connection (defined by $\frac{\partial j_n}{\partial z} = M_n^m j_m$)

$$M_n^m = \begin{cases} \dfrac{z^n}{2P(z)}\left((m+1)a_{m+1} + z^{-m-1}\sum\limits_{j=0}^{m} a_j z^j\right) & \text{for } 0 \le m < n, \\ \dfrac{z^n}{2P(z)}\left((m+1)a_{m+1} - \sum\limits_{j=0}^{2g-1-m} a_{m+1+j} z^j\right) & \text{for } n \le m \le 2g-1 \end{cases} . \tag{3.29}$$

One finds that the periods satisfy a similar recursive condition as (3.27). An analogous consideration can be performed for the $\bar{z}$-dependence of the periods. One finds that the integrability condition of the Picard–Fuchs systems is just the axisymmetric Laplace equation.

On the other hand, with the help of some boundary conditions (for instance at $|z| \to \infty$), the I_n can be uniquely determined from the above system (3.27). Thus the class of solutions discussed by Meinel and Neugebauer may be phrased in the following form: if an arbitrary solution of the Laplace equation is given, one can calculate the functions I_n with (3.27) and the boundary condition, and ends up with a solution to the Ernst equation of the form (3.20).

4 The singular structure of the Ernst potential

The construction of the solutions in the previous sections with the help of Theorem 2.1 also indicates where the resulting Ernst potential (3.20) may be singular: only at points $P_0 = -\mathrm{i}z$ where the conditions of Theorem 2.1 do not hold. Notice that these conditions are sufficient for the regularity of f at all other points. It may turn out though that the Ernst potential is

perfectly regular at points where Theorem 2.1 is not fulfilled, e.g. in the case of singularities that are pure gauge. We will therefore discuss all possible singular points of the solutions (3.20).

It is very helpful that this whole discussion can be performed on the Riemann surface $\mathcal{L}_H$ where one can use the powerful calculus on hyperelliptic surfaces. One does not have to work on the four sheeted surface $\hat{\mathcal{L}}$ whose introduction was necessary for the construction of the solutions, and which provides an understanding of the mathematical properties of the Ernst equation. Since we will work from now on on $\mathcal{L}_H$ only, unless otherwise noted, we will drop the index H at points $P \in \mathcal{L}_H$.

The possible singularities of f can be directly inferred from the potential in the form (3.20). The Ernst potential will be singular at the zeros of the denominator. It is possibly not regular at the points where P_0 is identical to the singularities of Ω or is on Γ. Critical points of a different kind are the branch points E_i and F_i. If P_0 coincides with these points, the Riemann surface $\mathcal{L}_H$ becomes singular. Something similar happens at the axis where the branch points P_0 and $\bar{P}_0$ coincide. This is a reminiscent of the singular behaviour of the three–dimensional Laplace operator on the axis in the axisymmetric case. The main aim of the following analysis is to single out a class of solutions that may be interesting in the context of boundary value problems for the Ernst equation that describe e.g. the exterior of a body of revolution. Thus we will not study the nature of the singularities (e.g. curvature singularities) but single out a large class of solutions where the Ernst potential is only discontinuous at a (closed) contour that could be identified with the surface of a body.

Zeros of the denominator Zeros of the denominator of (3.20) will lead to singularities in the spacetime. From condition IV of Theorem 2.1 it follows that these are just the points at which the matrix Φ cannot be normalized in the required way. This leads to the transcendental condition

$$\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\omega(\infty^-)+u+b)\neq 0, \tag{4.1}$$

if one wants to exclude these zeros of the denominator. We will show in the next section how the points at which $\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\omega(\infty^-)+u+b)=0$ can be found as the solution of a set of algebraic equations.

Essential singularities The integrals of the third kind occuring in Ω are nothing but a particular case of line integrals over contours with constant jump function $G(t)$. Therefore, we are left with the investigation of the integrals of the second kind at this point. Since the theta functions in (3.20) are regular as long as the Riemann surface $\mathcal{L}_H$ is, we are left with the exponent if (4.1) holds. For the behaviour of the exponent, we get the following

Proposition 4.1 *The Ernst potential (3.20) has an essential singularity at the points where P_0 coincides with the singularities of the integrals of the second kind on $\mathcal{L}_H$.*

Proof : The exponent has by construction an algebraic pole there and, therefore, the Ernst potential has an essential singularity. □

An essential singularity of the real part of the Ernst potential corresponds to a line singularity of the metric function U. In the context of exterior solutions for bodies of revolution we are interested in, there seems to be no situation where such a line singularity in a spacetime might be interesting.

Contours In the case that P_0 lies on a contour Γ_i but not on an endpoint of Γ_i, on the axis, or on one of the branch points E_i or F_i, it can be easily seen that the integral in the exponent as well as the b–periods u_i are bounded since G is Hölder–continuous, finite and non–zero on Γ. At the endpoints of the contours Γ_i, singularities may occur. The value of these integrals at the remaining points will however not be the same in general if the contour is approached from one or the other side. This can be seen from the following fact: The point P_0 is a branch point of $\mathcal{L}_H$. If it lies on the contour Γ, care has to be taken of the sign of the root whilst evaluating the integrals of the form $J_n = \int_\Gamma \ln G(\tau)\frac{\tau^n \mathrm{d}\tau}{\mu}$. The decisive factor is $\mu_0^2 = (K - P_0)(K - \bar{P}_0)$. We get for $K \in \Gamma$ with $K = K_1 + \mathrm{i}K_2$, and $K_1, K_2 \in \mathbb{R}$ for the imaginary part of μ_0,

$$\Im \mu_0 = \pm\frac{1}{\sqrt{2}}\mathrm{sgn}\left((K_1 - \zeta)K_2\right)\sqrt{|\mu_0^2| - \Re\left(\mu_0^2\right)}, \tag{4.2}$$

i.e. the sign of the imaginary part of μ depends on the sign of $K_1 - \zeta$. Thus the value of the integrals will in general not be the same whether the contour is approached from the interior or the exterior region. This reasoning does not work for points $K = K_0$ not on the axis ($K_2 \neq 0$) with $K_1 = 0$. There the imaginary part of μ_0^2 is zero which means that μ_0 is either purely imaginary or real in the vicinity of $P_0 = K_0$ depending on the sign of $K_2 - \zeta$. We conclude that the integrals over Γ with $P_0 \in \Gamma$ have the form $J = J^1 + \mathrm{sgn}(\epsilon)J^2$ (where the J^i are independent of ϵ which indicates if the contour is approached from the interior or the exterior) which implies that the limiting value of the Ernst potential calculated via (3.20) exists but depends on ϵ. Therefore, we have proven the following

Proposition 4.2 *Let P_0 lie on the contour Γ but not on the axis, on one of the branch points E_i or F_i, or at an endpoint of Γ_i. Then f will, in general, have a jump at Γ. The limiting value of f will exist and be Hölder–continuous there. f may be singular at the endpoints of the Γ_i.*

Thus the Ernst potential will be finite but discontinuous at a contour Γ_z in the (ρ, ζ)–plane given by $P_0 \in \Gamma$ which means that the solution to the vacuum equations will not be regular at a surface in the $(\rho,\ \zeta,\ \phi)$–space. If this surface is closed, it can possibly be identified with the surface of a body of revolution. The interior of the body is supposed to be filled with matter. Therefore the vacuum solution is only considered in the exterior (that contains $z = \infty$); it is not regular at the boundary to the matter region.

The axis The axis is a double point on the Riemann surface $\mathcal{L}_H$ since two branch points coincide. In this case, all quantities may be considered on the Riemann surface Σ' given by $\mu'^2 = \prod_{i=1}^{g}(\tau - E_i)(\tau - F_i)$, as was shown by Fay [34]. Let a prime denote here and in the following that the primed quantity is taken on Σ'. This surface is obtained from $\mathcal{L}_H$ by removing the cut $\left[P_0, \bar{P}_0\right]$. For the analysis of the axis, we will use a slightly different

cut–system than the one introduced in the previous section: we take a closed curve around $\left[P_0, \bar{P}_0\right]$ in the +–sheet as the cut a_g. All b–cuts shall begin at the cut $[E_1, F_1]$. The rest is unchanged. This implies for the characteristic of the theta function that it has the form

$$\begin{bmatrix} \alpha' & 1 \\ \beta' & \varepsilon \end{bmatrix} \tag{4.3}$$

where $\varepsilon = 0, 1$ and $\alpha'_i = 0$.

Since the expansions of all characteristic quantities of the Riemann surface are smooth in ρ except π_{gg} which is divergent as $\ln\rho$ for $\rho \to 0$, it follows that the Ernst potential has a regular expansion in ρ. For points P_0 not coinciding with real branch points or singularities of the exponent in (3.20), the Ernst potential is thus at least C^3. It follows from a theorem of Müller zum Hagen [35] that it is therefore analytic. Consequently it is sufficient to calculate the limiting case. If this is well defined, the Ernst potential is regular at these points of the axis. The differentials of the first kind for $\rho = 0$ turn out to be

$$\mathrm{d}\omega_i = \mathrm{d}\omega'_i, \quad i = 1, \dots, g-1, \quad \mathrm{d}\omega_g = -\mathrm{d}\omega'_{\zeta^+\zeta^-}, \tag{4.4}$$

where $\mathrm{d}\omega'_{\zeta^+\zeta^-}$ is the normalized differential of the third kind on Σ' with poles in ζ^+ and ζ^-. This implies for the b-periods

$$\pi_{ij} = \pi'_{ij}, \quad i, j = 1, \dots, g-1, \tag{4.5}$$

$$\pi_{ig} = -\int_{\zeta^-}^{\zeta^+} \mathrm{d}\omega'_i, \quad i = 1, \dots, g-1, \tag{4.6}$$

$$\pi_{gg} = 2\ln\rho + \text{reg. terms} \; . \tag{4.7}$$

Since π_{gg} diverges, the theta function will break down to a sum of two theta series on Σ' (in the case of genus $g = 1$, the surface Σ' has genus 0; the formula below can however be used if one replaces the theta function Θ' simply by a factor 1 which means that the axis potential can be expressed in terms of elementary functions in this case). We introduce integrals $\omega'(P)$ of the first kind with the property $\omega'(E_1) = 0$. The differential $\mathrm{d}\omega_{\infty^+\infty^-}$ on the axis becomes $\mathrm{d}\omega'_{\infty^+\infty^-}$. In the case of the contour integrals one has to observe that an additional factor $\operatorname{sgn}(K_1 - \zeta)$ in the notation of (4.2) occurs for the same reasons as there. Since the Abelian integrals of the second kind can be obtained from the integrals of the third kind by a limiting procedure, the same holds for these integrals and their b-periods. With the above settings, we obtain for (3.20)

$$f = \frac{\Theta'\left[\begin{smallmatrix}\alpha'\\ \beta'\end{smallmatrix}\right]\left(\omega'|_{\zeta^+}^{\infty^+} + u' + b'\right) + (-1)^\varepsilon \exp(-(\omega'_g(\infty^+) + u_g + b_g))\Theta'\left[\begin{smallmatrix}\alpha'\\ \beta'\end{smallmatrix}\right]\left(\omega'|_{\zeta^-}^{\infty^+} + u' + b'\right)}{\Theta'\left[\begin{smallmatrix}\alpha'\\ \beta'\end{smallmatrix}\right]\left(\omega'|_{\zeta^+}^{\infty^+} - u' - b'\right) + (-1)^\varepsilon \exp(-(\omega'_g(\infty^+) - u_g - b_g))\Theta'\left[\begin{smallmatrix}\alpha'\\ \beta'\end{smallmatrix}\right]\left(\omega'|_{\zeta^-}^{\infty^+} - u' - b'\right)}$$
$$\exp\left\{\Omega'|_{\infty^-}^{\infty^+} + \frac{1}{2\pi i}\int\limits_\Gamma \ln G(\tau)\mathrm{d}\omega'_{\infty^+\infty^-}(\tau) + b_g + u_g\right\}. \tag{4.8}$$

It can be seen from the above formula that the limiting value of f exists even if u_g diverges, provided (4.1) holds (f will be Hölder-continuous if u_g diverges). The Ernst potential will however have an essential singularity at the real singularities of Ω. We can summarize the above results.

Proposition 4.3 *Let condition (4.1) hold. Then the Ernst potential is regular on the axis except at the points where P_0 coincides with singularities of Ω, points of Γ, and branch points E_i, F_i.*

Remark : Though f is Hölder-continuous even if u_g diverges, it is interesting to note for the following when this will be the case. Obviously this can only happen at the real points of Γ. It can be seen however that u_g is always bounded at these points due to the reality condition, unless they are endpoints of Γ_i (this would lead to a conic singularity on the axis).

Real branch points If P_0 coincides with a real branch point E_i or F_i, this will be a triple point on $\mathcal{L}_H$. We get the following.

Proposition 4.4 *At points where P_0 coincides with the real branch point E_g, the limiting value of f exists. The Ernst potential is in general not differentiable there.*

Proof : We use the same cut system and the same notation as on the axis. Put $P_0 = E_g + x$ with $x = \delta e^{i\phi}$ and $\phi \in \mathbb{R}$, $\delta \in \mathbb{R}^+$. In order to expand f in powers of x and $\bar{x}$, one has to consider the a-periods, in particular

$$\begin{aligned} \oint_{a_g} \frac{d\tau}{\mu} &= \frac{4}{\sqrt{\bar{x}(F_g - E_g)}} \int_1^{\frac{1}{k}} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)(F_g - E_g + xt^2)}\mu''(F_g + xt^2)} \\ &= \frac{4}{\sqrt{\bar{x}(F_g - E_g)}\mu''(F_g)} (i\tilde{K}(k) + O(\delta)) \end{aligned} \tag{4.9}$$

where $k = e^{i\phi}$, where $\tilde{K}(k) = K(\sqrt{1-k^2})$ and $K(k)$ are the complete elliptic integrals of the first kind, and where $\mu''^2(\tau) = \prod_{i=1}^{g-1}(\tau - E_i)(\tau - F_i)$. It can be seen from (4.9) that the a–period has an expansion in powers of $\sqrt{\delta}$. The coefficients of the expansion in $\sqrt{x}$, and $\sqrt{\bar{x}}$ are ϕ–dependent, since the modul of the elliptic integrals is just $k = e^{i\phi}$. This implies for the differentials of the first kind $d\omega_i = d\omega_i' + O(\sqrt{\delta})$ for $i = 1, ..., g-1$, and $d\omega_g = d\omega_{g-1}$. Similarly $d\omega_{\infty^+\infty^-} = d\omega'_{\infty^+\infty^-} + O\left(\sqrt{\delta}\right)$. We get for the b-periods,

$$\pi_{gg} = -2\pi \frac{K(k)}{K_1(k)} \left(1 + O\left(\sqrt{\delta}\right)\right) \quad , \tag{4.10}$$

whereas $\pi_{(g-1)g} = O(\sqrt{\delta})$ and $\pi_{ij} = \pi'_{ij}$ for $i, j = 1, \ldots, g-1$ in the limit. Thus f can be expanded in $\sqrt{x}$ and $\sqrt{\bar{x}}$. Even in case that only integer powers in the expansion occur, the coefficients will be in general ϕ-dependent. Though the limiting value of f at $P_0 = E_g$ exists, f will in general not be differentiable at this point. □

This implies that the real branch points are singular points on the axis, possibly topological defects in the spacetime, see [14]. They should not occur in the context of exterior solutions for bodies of revolution we are interested in.

Non-real branch points If P_0 coincides with a branch point $E_g = \bar{F}_g$, the points E_g and F_g will be double points on $\mathcal{L}_H$. Thus the situation is similar to the one on the axis with the only exception that one ends up here with two double points. As on the axis, it is convenient to consider all quantities on a Riemann surface Σ'' given by $\mu''^2 = \prod_{i=1}^{g-1}(K-E_i)(K-F_i)$ where the double points are removed. All quantities with two primes are understood to be taken on this surface. We use the following cut system: let a_{g-1} be the circle around $[P_0, E_g]$, and a_g the circle around $\left[\bar{P}_0, F_g\right]$, both in the plus sheet. The remaining cuts are as on the axis, i.e. all b–cuts start at $[E_1, F_1]$. As on the axis, we get

Proposition 4.5 *Let (4.1) hold, and let $E_g = \bar{F}_g \notin \Gamma$. Then the Ernst potential is regular at the point $P_0 = E_g$. For $E_g \in \Gamma$, f is in general Hölder–continuous at $P_0 = E_g$.*

The proof is similar to the one on the axis and basically uses again results of Fay [34].

Proof : The case $g = 1$ may be checked directly with the help of the standard theory of elliptic theta functions (see e.g. [36]). For $g > 1$ with the cut system in use and $P_0 = E_g + x$, where x is chosen as in the case of the real branch points, the differentials of the first kind have a smooth expansion in x and $\bar{x}$. In contrast to the case of real branch points, the coefficients in the expansion are ϕ–independent. The differentials $\mathrm{d}\omega_i$ become in leading order the differentials of the first kind $\mathrm{d}\omega_i''$ on Σ''. The differential $\mathrm{d}\omega_{g-1}$ becomes in the limit the differential $-\mathrm{d}\omega''_{E_g^+E_g^-}$, and similar for $\mathrm{d}\omega_g$ at F_g. The differential of the third kind becomes $\mathrm{d}\omega_{\infty^+\infty^-} = \mathrm{d}\omega''_{\infty^+\infty^-}$. All these differentials have coefficients in the x and $\bar{x}$ expansion that contain Abelian integrals of the second kind with poles in $E_g^\pm$ and $F_g^\pm$ as may be checked by direct calculation. This implies for the b–periods that $\pi_{ij} = \pi''_{ij}$ for $i, j = 1, ..., g-2$ and

$$\begin{aligned}
\pi_{(g-1)(g-1)} &= \pi_{gg} = 2\ln\delta + ..., \\
\pi_{i(g-1)} &= -2\omega''(E_g^+), \\
\pi_{ig} &= -2\omega''(F_g^+),
\end{aligned} \tag{4.11}$$

whereas $\pi_{(g-1)g}$ is finite in the limit $\delta \to 0$. If $E_g \notin \Gamma$, the u_i as well as the Cauchy integral in the exponent have a smooth expansion in x and $\bar{x}$ with finite coefficients. The theorem of [35] then guarantees regularity if the limiting value that may be calculated as on the axis exists. The theta function on $\mathcal{L}_H$ breaks down to a sum of four theta functions on Σ'' times a multiplicative factor. If $E_g \in \Gamma$, the coefficients in the expansion of f in x and $\bar{x}$ will diverge which implies that f is possibly not differentiable there though the limiting value exists if (4.1) holds. □

5 Metric functions and ergospheres

In the previous sections we have made extensive use of the complete integrability of the Ernst equation to construct a large class of solutions. To discuss physical features of the resulting spacetimes however, it would be helpful to have expressions in closed form not only for the Ernst potential but for the metric functions, at least for the functions e^{2U} and a that can be expressed invariantly via the Killing vectors. It is a remarkable fact already noticed

by Korotkin [14] that the metric function a can be related to derivatives of the matrix Φ without solving the differential equation (2.2). In the following we will show that a theta identity of Fay [34] can be used to go one step further to obtain a formula for a that is free of derivatives. The same identity leads to a simplified expression for the metric function e^{2U} that can be directly used to identify ergospheres in the spacetime.

Fay's trisecant identity establishes a relation between four points $A_1, \ldots, A_4$ on a Riemann surface, in our case $\mathcal{L}_H$, in arbitrary position (see e.g. [37], [38]). Let x be an arbitrary g-dimensional vector. Then the following identity holds,

$$\Theta(x)\Theta\left(x+\int_{A_1}^{A_3}\mathrm{d}\omega+\int_{A_2}^{A_4}\mathrm{d}\omega\right)-\exp\left(\Omega_{A_1A_4}|_{A_2}^{A_3}\right)\Theta\left(x+\int_{A_2}^{A_3}\mathrm{d}\omega\right)\Theta\left(x+\int_{A_1}^{A_4}\mathrm{d}\omega\right)$$
$$-\exp\left(\Omega_{A_2A_4}|_{A_1}^{A_3}\right)\Theta\left(x+\int_{A_1}^{A_3}\mathrm{d}\omega\right)\Theta\left(x+\int_{A_2}^{A_4}\mathrm{d}\omega\right)=0\ , \tag{5.12}$$

where e.g. $\Omega_{A_1A_4}|_{A_2}^{A_3}$ denotes the integral of a normalized differential of the third kind with simple poles at A_1 respectively A_4 with residues $+1$ respectively -1 along a path from A_2 to A_3. For a geometric interpretation of this identity in terms of the Kummer variety see [37], [38], for an interpretation via generalized cross ratio functions see [24]. The strength of the above identity arises from the fact that it holds for points A_i in general position. By a suitable choice of these points, we obtain for the metric function e^{2U}, the real part of the Ernst potential,

$$e^{2U}=\frac{1}{2}\exp\left(\Omega_{\bar{P}_0\infty^-}|_{\infty^+}^{P_0}\right)\frac{\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(u+b)\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(u+b+\omega(\bar{P}_0))}{\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(u+b+\omega(\infty^-)+\omega(\bar{P}_0))\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(u+b+\omega(\infty^-))}e^{I} \tag{5.13}$$

where I denotes the integral in the exponent of (3.20). This formula makes it possible to identify directly the zeros of e^{2U} which give the ergospheres, the limiting surfaces of stationarity (inside these surfaces there can be no observer at rest with respect to spatial infinity). Since the exponent of the integral of the third kind in (5.13) in front of the fraction cannot vanish, the necessary condition for ergospheres is

$$\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(u+b)\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(u+b+\omega(\bar{P}_0))=0. \tag{5.14}$$

Defining the divisor A as the solution of the Jacobi inversion problem

$$\omega(A)-\omega(D)=u+b, \tag{5.15}$$

we find that an ergosphere can occur if P_0 or $\bar{P}_0$ are in A. It is however possible that the denominator of (5.13) vanishes at the same time which would imply a violation of (4.1) (and thus a singularity of the spacetime). Summing up we get

Proposition 5.1 *I. Let P_0 or $\bar{P}_0$ and ∞^- be in A for some P_0, then condition (4.1) is violated and the Ernst potential is singular at these points.*
II. Let P_0 or $\bar{P}_0$ but not ∞^- be in A for some P_0, then the real part of the Ernst potential vanishes at these points which describe an ergosphere.

The same formula can be used on the axis where we obtain for the metric function e^{2U} in the notation of (4.8)

$$\mathrm{e}^{2U} = \frac{1}{2}\exp\left(\Omega_{\infty^-\zeta^+}|_{\zeta^-}^{\infty^+}\right) \frac{\Theta'\begin{bmatrix}\alpha'\\ \beta'\end{bmatrix}^2(u'+b')}{\Theta'\begin{bmatrix}\alpha'\\ \beta'\end{bmatrix}^2(\omega'|_{\zeta^-}^{\infty^-}+u'+b') - \exp(2(\omega_g(\infty^-)+u_g+b_g))\Theta'\begin{bmatrix}\alpha'\\ \beta'\end{bmatrix}^2(\omega'|_{\zeta^+}^{\infty^-}+u'+b')} \quad . \tag{5.16}$$

The condition for an ergosphere to hit the axis is then

$$\Theta'\begin{bmatrix}\alpha'\\ \beta'\end{bmatrix}(u'+b') = 0 \ , \tag{5.17}$$

since the integral of the third kind in the exponent in front of the fraction cannot diverge for finite values of ζ. The interesting feature of this relation is that it is completely independent of the physical coordinates. This implies that if an ergosphere extends to the axis, this will be only possible if the metric function e^{2U} vanishes on the whole axis. In the case of the Kerr solution, the ergosphere touches the axis at the horizon. An interpretation of the fact that the whole axis would be singular in the present case is given in the next section where the above case is related to the ultrarelativistic limit in which the source of the gravitational field becomes so strong that it vanishes behind the horizon of the extreme Kerr metric.

The metric function a can be calculated from (2.13) if one uses the trisecant identity (5.12) in the limit that two points coincide. Using the trisecant identity several times, we get

$$(a-a_0)\mathrm{e}^{2U} = -\rho\left(\frac{\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0)\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\int_{\bar{P}_0}^{P_0}\mathrm{d}\omega\right)}{\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\int_{\infty^-}^{P_0}\mathrm{d}\omega\right)\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\int_{\bar{P}_0}^{\infty^-}\mathrm{d}\omega\right)}\times \frac{\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(u+b)\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(u+b+\int_{P_0}^{\infty^-}\mathrm{d}\omega+\int_{\bar{P}_0}^{\infty^-}\mathrm{d}\omega\right)}{\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(u+b+\int_{\bar{P}_0}^{\infty^-}\mathrm{d}\omega\right)\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(u+b+\int_{P_0}^{\infty^-}\mathrm{d}\omega\right)} - 1\right) \quad . \tag{5.18}$$

The constant a_0 can be obtained in a similar way from the condition that $a=0$ on the regular part of the axis (we assume here that the singularities in the exponent of (3.20) are situated in a compact region of the (ρ,ζ)–plane). Care has to be taken in the above formula that some of the terms in brackets explode as $1/\rho$ in the limit $\rho\to 0$. We get

$$a_0 = \frac{\mathrm{i}}{2}(D-\bar{D})\frac{\Theta'\begin{bmatrix}\alpha'\\ \beta'\end{bmatrix}(u'+b'+\int_{\infty^+}^{\infty^-}d\omega')\Theta'\begin{bmatrix}\alpha'\\ \beta'\end{bmatrix}(0)}{\Theta'\begin{bmatrix}\alpha'\\ \beta'\end{bmatrix}(\int_{\infty^+}^{\infty^-}d\omega')\Theta'\begin{bmatrix}\alpha'\\ \beta'\end{bmatrix}(u'+b')}e^{-I'} . \tag{5.19}$$

It can be seen from this formula that a_0 does not vanish if there are no singularities in the exponent ($I=u=b=0$) in which case $f=1$ which describes Minkowski spacetime. This reflects, as already noted, the fact that a_0 is a gauge dependent quantity. The metric function a however is gauge independent. In the above example of Minkowski spacetime, it will of course vanish in the used asymptotically non-rotating coordinates.

6 Asymptotic behaviour and equatorial symmetry

Since we are mainly interested in solutions to the Ernst equation that could describe the gravitational field outside a compact matter source, we will study the asymptotic behaviour (near spatial infinity) of the solutions (3.20). It is generally believed that the Ernst potentials of the corresponding spacetimes are regular except at the contour Γ_z in the (ρ, ζ)–plane which corresponds to the surface of the body, asymptotically flat and equatorially symmetric. We will investigate in the following whether it is possible to identify solutions with these properties in the class (3.20).

Asymptotic behaviour Asymptotic flatness implies that the Ernst potential is of the form $f = 1 - 2m/|z| + o(1/|z|)$ for $|z| \to \infty$ where m is a positive real constant. A complex m is related to a so called NUT-parameter that is comparable to a magnetic monopole.

The asymptotic properties of the solutions (3.20) can be read off at the axis. Notice that the $\mathrm{d}\omega_i'$ are independent of ζ. For $\mathrm{d}\omega_g$, we get

$$\mathrm{d}\omega_g = \mathrm{d}\omega'_{\infty^+\infty^-}\left(1 - \frac{1}{2\zeta}\sum_{i=1}^{g}(E_i + F_i)\right) + \frac{1}{\zeta}\mathrm{d}\omega'_{\infty^+,1} + o(1/\zeta) \tag{6.20}$$

where $\mathrm{d}\omega'_{\infty^+,1}$ is the differential of the second kind with a pole of second order at ∞^+. Furthermore it can be seen that $\exp(-\omega_g(\infty^+))$ is proportional to $1/\zeta$ for $\zeta \to \infty$. Thus we get

Proposition 6.1 *Let* $\lim_{\tau\to\infty} \tau \ln G(\tau) = 0$ *on all contours that go through* ∞^+ *or* ∞^- *and let* $\Theta'\begin{bmatrix}\alpha'\\ \beta'\end{bmatrix}(u' + b') \neq 0$. *Then* f *has the form* $f = 1 - 2m/\zeta$ *for* $\zeta \to \infty$ *where* m *is a complex constant.*

The proof of this proposition follows from (6.20) and (4.8).

Equatorial symmetry The fact that the mass is in general complex implies that the class we are considering here is too large if one wants to study only solutions that are asymptotically flat in the strong sense (m real). There is the belief that stationary axisymmetric spactimes describing isolated bodies in thermodynamical equilibrium are equatorially symmetric. This implies for the Ernst potential $f(-\zeta) = \bar{f}(\zeta)$. Solutions with this property always have a real mass due to the symmetry. It is therefore of special interest to single out equatorially symmetric solutions among those in (3.20). We get

Theorem 6.2 *Let* $\mathcal{L}_H$ *be a hyperelliptic surface of the form (2.21) with even genus* $g = 2s$ *and the property* $\mu(-K, -\zeta) = \mu(K, \zeta)$. *Let* Γ *be a piecewise smooth contour on* $\mathcal{L}_H$ *such that with* $P = (K, \mu(K)) \in \Gamma$ *also* $\bar{P} \in \Gamma$ *and* $(-K, \mu(K)) \in \Gamma$. *Let there be given a finite nonzero function* G *on* Γ *subject to* $G(\bar{P}) = \bar{G}(P) = G((-K, \mu(K)))$. *If* $(p, \mu(p))$ *is a singularity of* Ω, *the same should hold for* $(-p, \mu(-p))$. *Choose a cut sytem in a way that the cuts* a_i^1 *(*$i = 1, \ldots, s$*) encircle* $[-F_i, -E_i]$ *and* a_i^2 *encircle* $[E_i, F_i]$ *in the +–sheet (in the case of real branch points, the points are ordered in the way* $E_i < F_i < E_{i+1} < \ldots$*; points with the same real part are ordered in the way* $\Im(E_i) < \Im(F_i) < \Im(E_{i+1}) < \ldots$ *which implies that* $E_i \neq \bar{F}_i$ *in this special case).*

Then f is equatorially symmetric if the characteristics in the i–th position (any combination of the two cases is allowed) have the form

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{6.21}$$

Proof : The property $\mu(\zeta, K) = \mu(-\zeta, -K)$ on $\mathcal{L}_H$ makes it possible to express quantities on a surface with $\zeta = -\zeta_0$ in terms of the corresponding quantities on the surface with $\zeta = \zeta_0$. We have $a_i^1(-\zeta) = \tau a_i^2(\zeta)$ and $b_i^1(-\zeta) = \tau b_i^2(\zeta)$ where ζ and $-\zeta$ denote the surface on which the quantity is considered, and where τ is the anti–holomorphic involution on $\mathcal{L}_H$. Together with the symmetry properties of the Abelian integrals in the exponent of (3.20), this implies that the transformation $\zeta \to -\zeta$ acts as the complex conjugation together with a change of the upper index. Thus we have for the characteristics in (6.21) $f(-\zeta) = \bar{f}(\zeta)$. □

Remark : If the theta function contains only blocks of the form $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, the resulting f is just the complex conjugate of the Ernst potential built with the Riemann theta function. This means that the two cases in (6.21) are related through complex conjugation. It is however possible to combine any number of these two blocks in which case the Ernst potential cannot be simply reduced to the case of the Riemann theta function or its complex conjugate. In the case of a rotating body, complex conjugation of the Ernst potential only implies that the angular velocity of the body changes its sign.

The above results suggest that it is possible to identify a whole subclass of solutions among (3.20) that are asymptotically flat, regular except at a closed contour and equatorially symmetric, i.e. solutions that might describe the exterior of a rotating body and might be helpful in the construction of solutions to boundary value problems for the Ernst equation. We get

Theorem 6.3 *Let $\mathcal{L}_H$ be a regular hyperelliptic surface of even genus $g = 2s$ of the form (2.21) without real branch points. Let Γ be a closed, smooth contour on $\mathcal{L}_H$ such that with $P = (K, \mu(K)) \in \Gamma$ also $\bar{P} = (\bar{K}, \mu(\bar{K})) \in \Gamma$ and $(-K, \mu(K)) \in \Gamma$ and $E_i \notin \Gamma$. Let there be given a finite nonzero function G on Γ subject to $G(\bar{P}) = \bar{G}(P) = G((-K, \mu(K))$. Choose the characteristic $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ such that it consists of blocks of the form $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ as in theorem 6.2.*

Then

$$f(\rho, \zeta) = \frac{\Theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\omega(\infty^+) + u)}{\Theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\omega(\infty^+) - u)} \exp \left\{ \frac{1}{2\pi \mathrm{i}} \int_\Gamma \ln G(\tau) \mathrm{d}\omega_{\infty^+\infty^-}(\tau) \right\} \quad , \tag{6.22}$$

is

1. *a regular solution to the Ernst equation for $P_0 \notin \Gamma$ if condition (4.1) holds.*
2. *in general discontinuous at Γ_z given by $P_0 \in \Gamma$,*
3. *asymptotically ($|z| \to \infty$) given by $f = 1 - 2m/|z|$ where m is a finite real constant if $\Theta' \begin{bmatrix} \alpha' \\ \beta' \end{bmatrix} (u') \neq 0$,*
4. *equatorially symmetric.*

Proof : From (3.20) it can be seen that f is a solution to the Ernst equation. The regularity properties follow from the previous section. Asymptotic behaviour and equatorial symmetry follow from above. $\square$

Remark : The choice of this class is mainly due to regularity requirements. If all singularities like real branch points or the singularities of the Abelian integrals of the second kind lie within the contour Γ where the solution is not considered since the region is assumed to be filled with matter, they would not affect the vacuum region. However this would not enlarge the degrees of freedom (one real–valued function and a set of complex parameters) if one wants to solve boundary value problems.

We will discuss the common properties of the solutions in this subclass in the following.

Mass and Angular Momentum The asymptotic behaviour of the Ernst potential, $f = 1 - 2m/|z| - 2\mathrm{i}j/|z|^2 + \cdots$, follows already from the axis potential (4.8). The equatorial symmetry implies that m and j are real constants. In the following it is convenient to introduce rescaled coordinates $\tilde{z} = z/R$ where R is the radius of the smallest sphere that totally contains the contour Γ_z, and the dimensionless quantities $M = m/R$, $J = j/R^2$. This implies that the contour shrinks to a point in the limit $R \to 0$. If we use the differential operator D_{∞^+} we get for the ADM-mass with (4.8)

$$M = \frac{D_{\infty^+}\Theta\left[\begin{smallmatrix}\alpha'\\ \beta'\end{smallmatrix}\right](u')}{\Theta\left[\begin{smallmatrix}\alpha'\\ \beta'\end{smallmatrix}\right](u')} + \frac{1}{2p\mathrm{i}}\int_\Gamma \ln G \mathrm{d}\omega'_{1,\infty^+}. \tag{6.23}$$

A solution is of course only physically acceptable if the mass is positive. Similarly one can see that the angular momentum is proportional to $1/\Theta^2\left[\begin{smallmatrix}\alpha'\\ \beta'\end{smallmatrix}\right](u')$.

Minkowskian limit It is possible to parametrize a solution by the mass and the angular momentum. For $M << 1$ and $J << 1$, the solution is nearly Minkowskian. It can be directly seen that this limit is obtained for $G \to 1$. This implies that the solutions from above for $|G| \sim 1$ are in the regime of small gravitational fields. This is also the regime of the Newtonian limit if the solution has one.

Ultrarelativistic limit It can be seen from (6.23) that both the mass and the angular momentum diverge if $\Theta'\left[\begin{smallmatrix}\alpha'\\ \beta'\end{smallmatrix}\right](u') = 0$ but that M^2/J remains finite in this case. This suggests that this divergence is best understood as the limit $R \to 0$ as was already done in [39] for the case of the rigidly rotating dust disk: in this limit, the gravitational fields at the surface Γ_z of the matter source become so strong that it is hidden behind a horizon, i.e. its coordinate radius tends to zero. It is suggestive to consider this limit as the ultrarelativistic limit of the solution. For finite R, the Ernst potential is purely imaginary on the axis which means that the solution is no longer asymptotically flat. If one takes the limit $R \to 0$, one ends up with the Kerr solution in this case which gives further support to the interpretation of this limit as the ultrarelativistic limit (this interpretation is of course only consistent if the limit gives the extreme Kerr solution for which the horizon is given in the used coordinates

by $\rho = \zeta = 0$). If the limit $R \to 0$ is taken for $\rho = \zeta = 0$ with ρ/ρ_0 and ζ/ρ_0 finite (this corresponds to an observer on the contour that vanishes behind the horizon), the resulting solution will not be asymptotically flat. For an observer on Γ_z, the exterior region is in infinite geodesic distance, and thus completely decouples from the exterior. In the context of a boundary value problem this limit corresponds to a stability limit for the solution: if the boundary data reach a certain critical value, the solution will no longer be regular.

Thus solutions in (6.22) should be physically interesting in the parameter range $0 < M/R < \infty$. Notice that the solutions have an analytic continuation beyond the upper limit. It can be seen however from (6.23) that the mass changes its sign in this case since $\Theta \begin{bmatrix} \alpha' \\ \beta' \end{bmatrix} (u')$ has zeros of first order. Consequently these 'overextreme' solutions, that do not have a Newtonian limit, are probably not physically interesting, at least they are unacceptable in the region with negative mass.

7 Reduction of the Ernst potential

The explicit form of the solutions (6.22) in terms of theta functions has a number of advantages in contrast to the linear integral equations to which the solution of boundary or initial value problems can be reduced in the case of integrable non–linear evolution equations*: it is possible to identify physically interesting features as ergospheres or the ultrarelativistic limit explicitly. Since the theta functions are transcendental, final results normally can only be obtained numerically. It is however possible to address most features like the condition for the occurrence of ergospheres (5.14) directly without having to determine the Ernst potential numerically in the whole spacetime.

The numerical treatment of theta functions is comparatively simple since the exponential series converges rapidly due to the factor $\exp\left(\frac{1}{2}\pi_{ij}n_i n_j\right)$ where $\Re(\pi_{ij}) < 0$. It is however obvious that the numerics become more and more tedious the larger the genus g of the Riemann surface $\mathcal{L}_H$ is. Therefore it is an important question whether the Riemann surface can be reduced in physically interesting cases to surfaces of lower genus. Loosely speaking this is possible if there exists a special relation between the branch points (see Weierstrass' discussion of the case $g = 2$ which is referred to in [13] and references given therein). Since the branch points P_0, $\bar{P}_0$ are parametrized by the physical coordinates and can thus take on arbitrary complex values, such a reduction will only be possible at special points of the spacetime which will in general not be of special physical interest. A general reduction of the Riemann surface is possible if there exist non–trivial automorphisms on the surface. For the class of equatorially symmetric solutions discussed here, this is the case in the equatorial plane and on the axis. There the surfaces $\mathcal{L}_H$ and Σ' have defining equations $\mu(K)$ and $\mu'(K)$ which both depend only on K^2. Thus on both surfaces there is the involution T defined by $(K, \mu(K)) \to (-K, \mu(-K))$.

For the sake of simplicity, we will only discuss the characteristic $\alpha_i = \beta_i = 0$ and the case $E_i^1 = -\bar{E}_i^2$ (the general case can be inferred from the resulting relations without problems). We will concentrate on disks of radius ρ_0 since they are an interesting model for galaxies. The

*There are solutions in terms of theta functions for these equations, too. But as we have pointed out already, these solutions are always periodic or quasiperiodic. The solutions to the Ernst equation discussed here are however gauge equivalent to solutions to a linear integral equation as was shown in [17].

Ernst potential simplifies in the equatorially symmetric case at the disk where the boundary data are prescribed, what makes disks the most promising objects in the search for solutions to boundary value problems to the Ernst equation in closed form. We recall that the first solution of such a problem was found for the rigidly rotating dust disk [15].

In the equatorial plane ($\zeta = 0$), the surface $\mathcal{L}_H$ is then given by $\mu^2(K) = (K^2 + \rho^2)\prod_{i=1}^{s}(K^2 - E_i^2)(K^2 - \bar{E}_i^2)$. We cut the surface as before which implies $Ta_i^1 = a_1^2$, $Tb_i^1 = b_i^2$ and $\mathrm{d}\omega_i^1(TP) = -\mathrm{d}\omega_i^2(P)$ with $P \in \mathcal{L}_H$. The Riemann surface $\Sigma_1 = \mathcal{L}_H/T$ of genus s is then given by

$$\mu_1^2(x) = x(x+\rho^2)\prod_{i=1}^{s}(x - E_i^2)(x - \bar{E}_i^2) \ . \tag{7.24}$$

The holomorphic differentials dv_i in Σ_1 dual to (a_i, b_i) (the projection of the cuts on $\mathcal{L}_H$ onto Σ_1) follow from $\mathrm{d}v_i = \mathrm{d}\omega_i^1 - \mathrm{d}\omega_i^2$. The so called Prym differentials $\mathrm{d}w_i$ which change the sign under T are given by $\mathrm{d}w_i = \mathrm{d}\omega_i^1 + \mathrm{d}\omega_i^2$. They are holomorphic differentials on the Riemann surface Σ_2 of genus s with

$$\mu_2^2(y) = (y+\rho^2)\prod_{i=1}^{s}(y - E_i^2)(y - \bar{E}_i^2) \ , \tag{7.25}$$

which implies that the Prym variety is a Jacobi variety in this case. The Riemann matrix on $\mathcal{L}_H$ has the form

$$\Pi = \frac{1}{2}\begin{pmatrix} \Pi^1 + \Pi^2 & \Pi^2 - \Pi^1 \\ \Pi^2 - \Pi^1 & \Pi^1 + \Pi^2 \end{pmatrix} \ , \tag{7.26}$$

where the Π^i are the Riemann matrices on Σ^i respectively. The theta function on $\mathcal{L}_H$ thus factorizes into products of theta functions on the Σ_i,

$$\Theta(x_1|x_2, \Pi) = \sum_{\delta} \Theta\begin{bmatrix}\delta \\ 0\end{bmatrix}(x_1 + x_2; 2\Pi^2)\Theta\begin{bmatrix}\delta \\ 0\end{bmatrix}(x_1 - x_2; 2\Pi^1) \ , \tag{7.27}$$

where each component of the s-dimensional vector δ takes the values $0, 1$. Thus the theta function on the surface of genus $2s$ can be expressed via theta functions on surfaces of genus s.

In the case of the Ernst potential (6.22), further simplifications follow from the fact that ∞ is a branch point of Σ_2. For the contour integrals u, we obtain for disks

$$u_v = \frac{1}{\pi \mathrm{i}}\int_{\Gamma_v} \ln G \mathrm{d}v, \quad u_w = \mathrm{sgn}\zeta \frac{1}{\pi \mathrm{i}}\int_{\Gamma_w} \ln G \mathrm{d}w \ , \tag{7.28}$$

where Γ_v is the contour in the $+$-sheet of Σ_1 between 0 and $-\rho^2$ along the real axis, and Γ_w is the part of the real axis in the upper sheet of Σ_2 between $-\infty$ and $-\rho^2$. The formula for u_w shows that it does matter whether the equatorial plane is approached from the upper or the lower side (the Ernst potential is not regular at the disk). For $\rho > \rho_0$, we have $u_w = 0$ ($G = 1$ in the exterior of the disk). Similarly we get for the integral in the exponent of (6.22)

$$I_v \doteq \frac{1}{2\pi \mathrm{i}}\int_{\Gamma} \ln G \mathrm{d}\omega_{\infty^+\infty^-} = \frac{1}{2\pi \mathrm{i}}\int_{\Gamma_v} \ln G \mathrm{d}v_{\infty^+\infty^-} \ . \tag{7.29}$$

Summing up we can write the Ernst potential in the equatorial plane in the form

$$f = \frac{\sum_\delta \Theta \begin{bmatrix} \delta \\ 0 \end{bmatrix} (v(\infty^+) + u_v; 2\Pi^1) \Theta \begin{bmatrix} \delta \\ \beta \end{bmatrix} (u_w; 2\Pi^2)}{\sum_\delta \Theta \begin{bmatrix} \delta \\ 0 \end{bmatrix} (v(\infty^-) + u_v; 2\Pi^1) \Theta \begin{bmatrix} \delta \\ \beta \end{bmatrix} (u_w; 2\Pi^2)} \mathrm{e}^{I_v} \ , \tag{7.30}$$

where $\beta_i = 1$, and where $v(P) = \int_{-\rho^2}^{P} dv$. The reality properties of the above theta functions imply together with (7.28) the condition for equatorial symmetry $f(-\zeta) = \bar{f}(\zeta)$. Thus the imaginary part of f jumps at the disk. For $\rho > \rho_0$ (where $u_w = 0$), only the terms with even characteristics in (7.30) will survive which leads to a real Ernst potential. This implies that the Ernst potential is regular outside the disk as it should be. The formula (7.30) can also be used to determine asymptotic quantities as angular momentum and ADM-mass in the limit of $\rho \to \infty$ as was done previously on the axis.

A similar reduction as in the equatorial plane is possible on the axis. There the Riemann surface Σ' also has the involution T which makes it possible to factorize the surface into the surfaces Σ_1' and Σ_2 where the Σ_i are as above and where Σ_1' is Σ_1 with the cut $[0, -\rho^2]$ removed. Thus the theta function Θ' on the surface Σ' of genus $2s-1$ can be expressed via theta functions on surfaces of genus $s-1$ and s respectively. In the case $g = 2$, this will not lower the genus of the Riemann surfaces under consideration on the axis. We will not give the Ernst potential on the axis since it will not be used here (the formula is helpful if one wants to calculate the multipole moments on the axis for higher genus).

8 The case $g = 2$

The simplest non-static solutions within the class (6.22) are of genus 2 since solutions of genus 0 belong to the Weyl-class. Interestingly the first solution to a physically relevant boundary value problem, the rigidly rotating dust disk [15] with dust parameter $\nu = 2\Omega^2\rho_0^2\mathrm{e}^{-2V_0}$ where Ω is the angular velocity in the disk, and $\mathrm{e}^{-V_0} - 1$ is the central redshift and radius ρ_0, belongs to this subclass. There the characteristic is $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, the branch points are given by $E = \sqrt{\mathrm{i}/\nu - 1}$, and the function G has the form $G = \left(\sqrt{1 + \nu^2(K^2+1)^2} + \nu(K^2+1)\right)^2$ where we have used dimensionless coordinates ρ/ρ_0 and ζ/ρ_0. Therefore we will discuss the case $g = 2$ as an example in more detail.

Solutions (6.22) of genus 2 will be regular except at the contour Γ_z if $\Theta(\omega(\infty^-) + u) \neq 0$. This is equivalent to the condition that the divisor A, defined by the Jacobi inversion problem $\omega(A) - \omega(D) = u$, does not contain both ∞^- and X where $X = P_0$ or $X = \bar{P}_0$. This implies that the equations

$$\int_{-E}^{\infty^-} \frac{\mathrm{d}\tau}{\mu(\tau)} + \int_{\bar{E}}^{X} \frac{\mathrm{d}\tau}{\mu(\tau)} = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\ln G \mathrm{d}\tau}{\mu(\tau)} \tag{8.31}$$

$$\int_{-E}^{\infty^-} \frac{\tau\mathrm{d}\tau}{\mu(\tau)} + \int_{\bar{E}}^{X} \frac{\tau\mathrm{d}\tau}{\mu(\tau)} = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\ln G\tau \mathrm{d}\tau}{\mu(\tau)} \tag{8.32}$$

must not hold simultaneously for all ρ and ζ in the vacuum region. Since the limits of the integration are fixed, this inversion problem will in general not have a solution. The equations

constitute a relation between the physical parameters that characterize the solution. In the case of the dust disk, this is the parameter ν and the radius ρ_0. This implies that the above condition will determine the allowed parameter range for ν for which there are no further singularities in the whole spacetime except the disk.

The condition for ergospheres can be obtained from a similar system of equations as above: simply replace ∞^- in (8.31) and (8.32) by a point on $\mathcal{L}_H$ which must not be ∞^- or X but is otherwise arbitrary. Then an ergosphere occurs if both equations hold simultaneously which gives in the above example the values for ν for which there exists a non–empty set of points ρ and ζ, the ergosphere. Since one of the points in the divisor A is in this case essentially arbitrary, the conditions for ergospheres will be satisfied much more frequently than the condition for a singularity where both points of A are prescribed.

The ergosphere has only common points with the axis in the ultrarelativistic limit which is given by $\vartheta_4(u_1')=0$ (we use the notation for elliptic theta functions of [36]). This condition is equivalent to

$$\frac{1}{2\pi \mathrm{i}}\int_\Gamma \frac{\ln G \mathrm{d}\tau}{\mu'(\tau)} = (2n+1)\int_{-E}^{E}\frac{\mathrm{d}\tau}{\mu'(\tau)} \tag{8.33}$$

where $n=0,1,2,\ldots$. Since this relation is independent of the physical coordinates, it determines the values for ν at which the ADM-mass diverges. Together with the conditions (8.31) and (8.32), this determines the allowed parameter range for ν: the absence of singularities except the disk and the fact that the mass shall vary between 0 and ∞ (the ultrarelativistic limit).

In the case $g=2$, the potential in the equatorial plane and on the axis can be expressed in terms of elliptic theta functions. The formula for the axis reads

$$f(\rho=0,\zeta) = \frac{\vartheta_4(\omega_1'|_{\zeta^+}^{\infty^+}+u_1')\vartheta_1(\omega_1'|_{\zeta^-}^{\infty^+}) + \exp(-u_2)\vartheta_4(\omega_1'|_{\zeta^-}^{\infty^+}+u_1')\vartheta_1(\omega_1'|_{\zeta^+}^{\infty^+})}{\vartheta_4(\omega_1'|_{\zeta^+}^{\infty^+}-u_1')\vartheta_1(\omega_1'|_{\zeta^-}^{\infty^+}) + \exp(u_2)\vartheta_4(\omega_1'|_{\zeta^-}^{\infty^+}-u_1')\vartheta_1(\omega_1'|_{\zeta^+}^{\infty^+})}$$
$$\exp\left\{\frac{1}{2\pi\mathrm{i}}\int\limits_\Gamma \ln G(\tau)\mathrm{d}\omega'_{\infty^+\infty^-}(\tau) + u_2\right\}. \tag{8.34}$$

In the equatorial plane we have

$$\bar{f} = \frac{\vartheta_3(v+u_v;2\Pi^1)\vartheta_4(u_w;2\Pi^2) - \vartheta_2(v+u_v;2\Pi^1)\vartheta_1(u_w;2\Pi^2)}{\vartheta_3(u_v-v;2\Pi^1)\vartheta_4(u_w;2\Pi^2) + \vartheta_2(u_v-v;2\Pi^1)\vartheta_1(u_w;2\Pi^2)}e^{I_v} \tag{8.35}$$

where $v=\int_{-\rho^2}^{\infty^+}\mathrm{d}v$. For $\rho>\rho_0$, the exterior of the disk, we get

$$f=\bar{f} = \frac{\vartheta_3(v+u_v;2\Pi^1)}{\vartheta_3(u_v-v;2\Pi^1)}e^{I_v}. \tag{8.36}$$

In both cases the formulae for a different characteristic are obtained by complex conjugation.

The above relations illustrate that important features of the solutions of genus 2 can be discussed with the help of the standard elliptic theory. It is thus an interesting question which boundary value problems lead to solutions within this subclass.

9 Outlook

In this paper, it was shown that it is possible to identify within a class of hyperelliptic solutions to the Ernst equation a subclass whose solutions could describe the exterior of a body of revolution: they are asymptotically flat, equatorially symmetric and regular except at the surface of the body. This subclass could consequently be interesting in the context of boundary value problems for the Ernst equation. There the boundary data are either induced by an interior solution or a surfacelike matter distribution as in the case of the dust disk: in the general case, one would have to fix two real functions at the boundary in order to satisfy the boundary conditions. Within the class considered here, one has the freedom to choose one real valued function (G) and a set of free parameters E_i, the branch points of the Riemann surface. Whether the subclass discussed here can actually be used to solve boundary value problems, and when these degrees of freedom will be indeed sufficient is an open question. However it is remarkable that it is possible to identify a whole generic subclass of regular equatorially symmetric solutions. This gives reasonable hope that further solutions to physically interesting boundary value problems may be found within the class of hyperelliptic solutions to the Ernst equation.

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